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Published in:
Numerical algorithms

DOI:
10.1007/s11075-017-0289-x

Noname manuscript No.
(will be inserted by the editor)

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Received: date / Accepted: date

Abstract The local convergence of generalized Mann iteration is investigated in the setting of a real Hilbert space. As application we obtain estimations of the local radius of convergence for some known iterative methods. Numerical experiments are presented showing the efficiency of the proposed estimates. In the case of One point Ezquerro-Hernandez method (J.A.Ezquerro, M.A.Hernandez, An optimization of Chebyshev's method, Journal of Complexity, 25 (2009) 343-361) the proposed procedure gives radii which are very close to the maximum local convergence radii.

Keywords Fixed point · Iterative method · Mann iteration · Local convergence · Local convergence radius

1 Introduction

Let $\mathcal{H}$ be a real Hilbert space (scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$), C an open subset of $\mathcal{H}$ and $T : C \to \mathcal{H}$ a nonlinear mapping. In this paper we are concerned with the problem of estimation of the local radius of convergence for the generalized Mann iteration. Recall that the generalized Mann iteration is defined by [1]

$$x_{n+1} = (I - D_n)x_n + D_nT(x_n), \tag{1.1}$$

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where I is the identity mapping and $\{D_n\} \subset \mathcal{L}(\mathcal{H})$ is the *generalized control sequence*. Recall also that a mapping T is said to be generalized demicontractive with respect to the *control linear mapping* $D : C \to \mathcal{L}(C)$ if

$$\langle D_x(x - T(x)), x - p\rangle \geq \lambda \|D_x(x - T(x))\|^2, \ \forall x \in C, p \in Fix(T),$$

where λ is a positive number. The generalized control sequence is a sequence of linear bounded mappings, usually defined by a function of x, $D_x = D(x)$ (we will use the notation $D_n = D_{x_n}$), or it can be defined recursively as a mapping depending on x_n and D_{n-1}. In fact (1.1) is equivalent to $x_{n+1} = x_n - D_n F(x_n)$ where $F(x) = x - T(x)$. Iterations of this type have been extensively investigated over the years, especially the case $D_n = B_n^{-1}$; however, considering it as Mann iteration some new results can be obtained.

Various known iterative methods for iterative approximation of the solutions of the equation $F(x) = 0$ are particular cases of (1.1). The following two cases will be considered in the sequel:

1. The *Newton method*, obtained from (1.1) by choosing $T(x) = x - F(x)$ and $D_x = F'(x)^{-1}$.

2. The *One point Ezquerro-Hernandez method*, obtained from (1.1) by choosing $T(x) = x - F(x) - F(x - F'(x)^{-1}F(x))$ and $D_x = F'(x)^{-1}$.

Some efforts have been made to obtain improved values of the local radius of convergence, especially in the case of Newton method or its variants. However "... effective, computable estimates for convergence radii are rarely available" [2]. Among the oldest known results we could mention those given by Vertgeim (1956) [3], Rall (1974) [4], Rheinboldt (1975) [2], Traub and Wozniakowski (1979) [5], Smale (1997) [6]. Relatively recent results (in the last decade) on these topics were communicated by Argyros [7–9], Ferreira [10], Hernandez-Veron and Romero [11], Ren [12] and Wang [13]. Determining effective and computable estimates for the local convergence radius is challenging and we aim to make a contribution in this sense.

In this paper we propose a general procedure (algorithm) for the estimation of the local radius of convergence for generalized Mann iteration (1.1). The procedure proves to be *efficient*, i.e., it provides a radius close to the maximal one and it is satisfactory computable. As applications we obtain the local radius of convergence for the two known methods mentioned above. It is worthwhile to mention that in the case of One point Ezquerro-Hernandez method, the experiments show that our algorithm gives local radius of convergence very close (or even identical) to the maximum radius of convergence.

2 Preliminary lemmas

We shall suppose throughout the paper that the set of fixed points of T is nonempty, $Fix(T) \neq \emptyset$.

Lemma 1 *Let $T : C \to \mathcal{H}$ be a Fréchet differentiable mapping on C. Then for given points $x, p \in C$ there exists a linear mapping $R_{x,p}$ (which depends on x, p), such that* *(i)* $T(x) - T(p) = (T'(x) + R_{x,p})(x - p)$;

(ii) for any $\varepsilon > 0$ *there exists* $r_\varepsilon > 0$ *such that if* $x \in B(p, r_\varepsilon) = \{x| \; \|x - p\| \leq r_\varepsilon\}$ *then* $\|R_{x,p}\| \leq \varepsilon$.

The proof is straightforward by defining $R_{x,p}$ as

$$R_{x,p}(u) = \frac{\langle x - p, u\rangle}{\|x - p\|^2}(T(x) - T(p) - T'(x)(x - p)).$$

Lemma 2 *[14] Let p be a fixed point of T. Suppose that T is Fréchet differentiable on C and that $I - T'(p)$ is invertible,* $\eta = \|(I - T'(p))^{-1}\|$. *Let c be such that $0 < c < \eta^{-1}$ and let r_c be defined in Lemma 1 for $\varepsilon = c$. Then*

$$\|x - p\| \leq \beta\|x - T(x)\|, \; \forall x \in B(p, r_c), \tag{2.1}$$

where $\beta = \frac{\eta}{1 - c\eta}$.

The condition (2.1), called *quasi expansivity*, was considered in a recent paper [15], in order to prove the strong convergence of the Mann iteration for strongly demicontractive mappings. It is easy to see that (2.1) implies $\|T(x) - p\| \geq \frac{1-\beta}{\beta}\|x - p\|$ which justifies the terminology *quasi-expansive*. It is also obvious that the set of fixed points of a mapping T which satisfies (2.1) consists of a unique element p in $B(p, r_c)$.

Condition (2.1) is similar but stronger to the following condition:

$$\|x - T(x)\| \geq \alpha \inf_{p \in Fix(T)} \|x - p\|, \; \forall x \in C,$$

where $0 < \alpha < 1$, which is considered in [16,17] as an additional condition to prove strong convergence of the Mann iteration for nonexpansive (quasi-nonexpansive) mappings in Banach spaces.

3 Local convergence

The Theorem 1 in this section provides conditions under which the iteration (1.1) converges weakly/strongly to a fixed point of T; its proof is similar to the proof of Theorem 1 from [18] and therefore omitted.

Theorem 1 *Let $T : C \to \mathcal{H}$ be a nonlinear mapping with a nonempty set of fixed points, $Fix(T) \neq \emptyset$. Let p be a fixed point of T and r such that $B(p, r) \subset C$. Suppose the following conditions are satisfied:*

(i) T is demiclosed at zero on $B(p, r)$;

(ii) D_x is invertible and $\|D_x^{-1}\| \leq M$, $\forall x \in B(p, r)$;

(iii) T is generalized demicontractive with $\lambda > 0.5$ for $\forall x \in B(p, r)$ and $\forall p \in Fix(T)$.

Then the sequence generated by the generalized Mann iteration (1.1) with starting point in $B(p, r)$ remains in $B(p, r)$ and converges weakly to a fixed point of T. If, in addition, T satisfies the quasi-expansivity condition (2.1) on $B(p, r)$, then the sequence converges strongly to the unique fixed point of T in $B(p, r)$.

In the sequel we apply Theorem 1 to prove local convergence of the two methods considered in Introduction.

For the Newton method, taking $T(x) = x - F(x)$ and $D_x = F'(x)^{-1}$, we get:

Corollary 1 *Let $F : C \to \mathcal{H}$ be a nonlinear mapping, where C is an open subset of $\mathcal{H}$. Suppose that F is Fréchet differentiable on C, that there exists $F'(x)^{-1}$, $\|F'(x)^{-1}\| \leq \beta$, $\forall x \in C$ and that F' is L-Lipschitz continuous on C. Then the set of solutions of $F(x) = 0$ is composed of isolated points and the Newton method converges locally to some solution.*

Proof Let p be a solution of $F(x) = 0$ (or a fixed point of T). From Lemma 2 it follows that T is quasi-expansive on some $B(p, r_c)$ and, therefore, p is the unique fixed point of T in $B(p, r_c)$. Let $r_\eta = 2\eta/(\beta L)$, where $\eta < \sqrt{5} - 2$ and $r \leq \min\{r_c, r_\eta\}$. We can suppose that $B(p, r) \subset C$.

Using the mean value theorem, we have for any $x \in B(p, r)$

$$\begin{aligned} D_x(x - T(x)) &= F'(x)^{-1}F(x) = F'(x)^{-1}(F(x) - F(p)) \\ &= F'(x)^{-1}\left(\textstyle\int_0^1 F'(p + t(x - p))dt\right)(x - p) \\ &= \left[I - F'(x)^{-1}\left(F'(x) - \textstyle\int_0^1 F'(p + t(x - p))dt\right)\right](x - p). \end{aligned}$$

Using the notation

$$\Delta_x = F'(x)^{-1}\left(F'(x) - \int_0^1 F'(p + t(x - p))dt\right),$$

the condition of generalized demicontractivity becomes

$$\langle (I - \Delta_x)(x - p), x - p\rangle \geq \lambda \|(I - \Delta_x(x - p)\|^2. \qquad (3.1)$$

We have

$$\begin{aligned} \|\Delta_x\| &\leq \|F'(x)^{-1}\| \textstyle\int_0^1 \|F'(x) - F'(p + t(x - p))\|dt \\ &\leq \beta L \textstyle\int_0^1 \|(1 - t)(x - p)\|dt \leq \frac{1}{2}\beta r L \leq \eta, \ x \in B(p, r). \end{aligned}$$

Now, from $0 < \eta < \sqrt{5} - 2$ it results $0.5 < (1 - \eta)/(1 + \eta)^2 < 1$. Let λ be such that

$$0.5 < \lambda < \frac{1 - \eta}{(1 + \eta)^2}.$$

Consider the quadratic polynomial

$$P(t) = \lambda t^2 + (2\lambda + 1)t - 1 + \lambda.$$

The largest solution of P is $s(\lambda) = (-2\lambda - 1 + \sqrt{8\lambda + 1})/(2\lambda)$. Now because $\eta < s(\lambda)$ and $P(0) < 0$, for any $0 < t \leq \eta$ it results $P(t) \leq 0$. As $\|\Delta_x\| \leq \eta$ we have that $P(\|\Delta_x\|) \leq 0$ which is equivalent to

$$1 - \|\Delta_x\| \geq \lambda(1 + \|\Delta_x\|)^2, \ \forall x \in B(p, r).$$

For any $\|y\| = 1$ we have

$$\begin{aligned}\langle (I - \Delta_x)y, y\rangle &= 1 - \langle \Delta_x y, y\rangle \geq 1 - \|\Delta_x\| \\ &\geq \lambda(1 + \|\Delta_x\|)^2 \geq \lambda\|I - \Delta_x\|^2 \geq \lambda\|(I - \Delta_x)y\|^2.\end{aligned}$$

Taking $y = (x-p)/\|x-p\|$ we obtain (3.1) i.e., $T(x) = x - F(x)$ is generalized demicontractive with respect to $F'(x)^{-1}$ and with $\lambda > 0.5$ on $B(p, r)$, which is the condition (iii) of Theorem 1. The condition (i) is obviously satisfied and condition (ii) results from L-Lipschitz continuity of F' and $D_x = F'(x)^{-1}$.

For One point Ezquerro-Hernandez method, taking $T(x) = x - F(x) - F(x - F'(x)^{-1}F(x))$ and $D_x = F'(x)^{-1}$, we get:

Corollary 2 *Suppose that F satisfies the conditions of Corollary 1. Then the set of solutions of $F(x) = 0$ is composed of isolated points and the One point Ezquerro-Hernandez method converges locally to some solution.*

Proof The proof follows the same lines as the proof of Corollary 1. The radius r_η is defined now as the (unique) real positive root of the equations

$$\frac{\beta^2 L^2}{2}\left(1 + \frac{\beta L}{4} r\right) r^2 - \eta = 0,\ \eta < \sqrt{5} - 2, \tag{3.2}$$

and $r = \min\{r_c, r_\eta\}$. Using the notations: $w = x - F'(x)^{-1}F(x)$, $I_x = \int_0^1 F'(p + t(x - p))dt$, $I_w = \int_0^1 F'(p + t(w - p))dt$, $\Delta_x = F'(x)^{-1}(F'(x) - I_w)F'(x)^{-1}(F'(x) - I_x)$, we get for $x \in B(p, r)$

$$D_x(x - T(x)) = (I - \Delta_x)(x - p)$$

and the condition of generalized demicontractivity becomes (3.1). We can estimate the upper bond of $\|\Delta_x\|$ as

$$\|\Delta_x\| \leq \frac{\beta^2 L^2}{2}\left(1 + \frac{\beta L}{4} r\right) r^2.$$

Now, if r is less than the positive root of (3.2) then $\|\Delta_x\| < \eta$. The rest of the proof follows as the proof of Corollary 1.

Following the same type of reasonings we can obtain conditions to local convergence and values for local radii of convergence for other iterative methods. For example, taking $D_x = F'(x_0)^{-1}$ and $T(x) = x - F(x)$ the generalized Mann iteration reduces to Modified Newton method, $x_{n+1} = x_n - F'(x_0)^{-1}F(x_n)$. Supposing that all conditions of Corollary 1 are satisfied, the convergence of this method is assured if x_0 belongs at the ball $B(p, r)$, where $r \leq 3\eta/(2\beta L)$.

The Picard iteration (successive approximations) can be also obtained from (1.1) by taking $D_x = I$ and $T(x) = F(x)$. In this case $\Delta_x = \int_0^1 F'(p - t(x - p))dt$ and the condition $\|\Delta_x\| \leq \eta$ is satisfied if $\|F'(x)\| \leq \eta = \sqrt{5} - 2$ on some ball on which the other conditions of Corollary 1 are fulfilled.

These corollaries show that the conditions of Theorem 1 are satisfied for a sufficiently large class of mappings; they do not provide algorithms that efficiently compute convergence radii. For example, the estimation given in Corollary 1 ($r = 2\eta/\beta L \approx 0.472/\beta L$) is less that the value proposed by Rheinboldt [2] ($r = 2/3\beta L \approx 0.667/\beta L$).

In finite dimensional spaces the condition of quasi-expansivity is superfluous; the first three conditions of Theorem 1 are sufficient for the convergence of the generalized Mann iteration. Therefore, in finite dimensional spaces, supposing that conditions (i), (ii) of Theorem 1 are fulfilled, we can develop the following algorithm to estimate the local radius of convergence:

Find the largest value for r such that

$$m = \min_{x \in B(p,r)} \frac{\langle D_x(x - T(x)), x - p\rangle}{\|D_x(x - T(x))\|^2} \qquad (4.1),$$

and $m > 0.5$.

This procedure involves the following main processing:

1. Apply a search line algorithm (for example of the type half-step algorithm) on the positive real axis to find the largest value for r;
2. At every step of 1 solve the constraint optimization problem (4.1) and verify the condition $m > 0.5$.

Several numerical experiments in one and several dimensions were performed to validate this method. It is worthwhile to underline that the values obtained by the proposed algorithm are, to some extent, larger than those given by other methods (Experiment 4), and, in some cases, our procedure gives local radii of convergence very close to the maximum ones (Experiments 1,2,3).

4 Numerical experiments

This section is devoted to numerical experiments to evaluate the efficiency of the proposed procedure. The obtained radii are compared (numerically or graphically) with the maximum radii of convergence. In our experiments the maximum radius was computed by directly checking the convergence of the iteration process starting from all points of a given net of points. The attraction basin (hence the maximum convergence radius) computed in this way has only relative worth. Nevertheless, this method provides significant information about the attraction basins, and the efficiency of proposed algorithm can be accurately evaluated.

Experiment 1

We have computed the local radius of convergence with the proposed algorithm corresponding to the two considered methods, and for a number of real functions. In all these examples the radii corresponding to Newton method are close to the maximum radii, and the radii corresponding to One point Ezquerro-Hernandez method coincide with the maximum radii. For example, in the case of function $f(x) = 0.5x^2 + cos(x)$ with $p = 1.04855836...$

the estimates (computed with seven decimal digits) are: for Newton method, $r = 0.3501329$ (maximum radius, $r_m = 1.0485583$); for One point Ezquerro-Hernandez method, $r = 0.2549290$ (maximum radius, $r_m = 0.2549290$).

Experiment 2

We applied the proposed algorithm to estimate the local radius of convergence for Newton method, for Picard iteration, and for One point Ezquerro-Hernandez method and for a number of mappings in several variables. For the following test mapping (we will refer to it as Ex.1):

$$F(x) = \begin{pmatrix} 0.8x_1 - cos(x_1) + x_2^2 + 1 \\ x_1^3 + 0.8x_2 \end{pmatrix},$$

and the fixed point $p = (0, 0)^T$ the results are given in Figure 1.

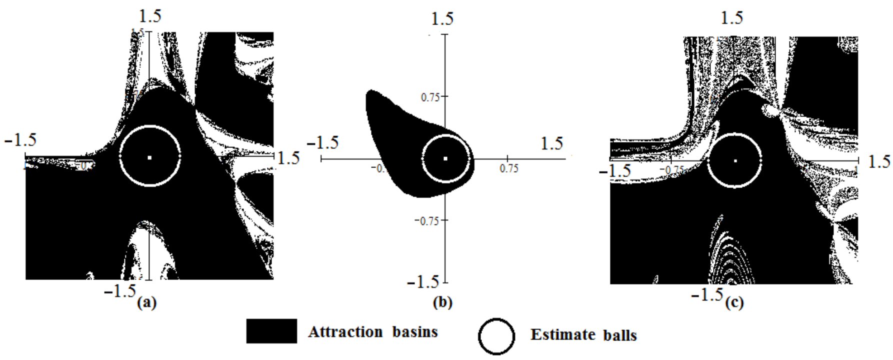

Fig. 1 Estimations of local convergence radii with proposed algorithm:
(a) For Newton method; (b) For Picard iteration; (c) For One point Ezquerro-Hernandez method

The black areas represent the whole attraction basins and the white circles the local convergence balls. It can be seen that the proposed algorithm gives local radii of convergence with satisfactory accuracy in the all cases. In the case of One point Ezquerro-Hernandez method the estimate are very close to the maximum radius of convergence or even coincide with them. For example, the estimate (with four decimal digits) is $r = 0.3192$ and it coincides with the maximum radius. Indeed, for any starting point $\|x_0\| \leq 0.3192$, the One point Ezquerro-Hernandez method converges to $p = (0, 0)^T$; for $x_0 = (-0.2850, 0.1441)$, $\|x_0\| = 0.3193$ the method fails to converge.

Experiment 3

We applied our algorithm to the Newton method and to the One point Ezquerro-Hernandez method for the complex function $f(z) = z^3 - 1$. In the case of Newton method the attraction basins corresponding to the three solutions of the equation $z^3 - 1 = 0$ are well known and the reunion of these basins has an interesting fractal structure. In Figure 2 it is drawn only the attraction basin corresponding to the solution $z = 0$. The same structure proves to be maintained in the case of the One point Ezquerro-Hernandez method. We could find the same conclusion as we found in Experiments 1 and 2: our algorithm gives radii of convergence close to maximum radii for the Newton method, and very close to the maximum ones (or even coincide with them) for the One point Ezquerro-Hernandez method.

The results are presented in Figure 2.

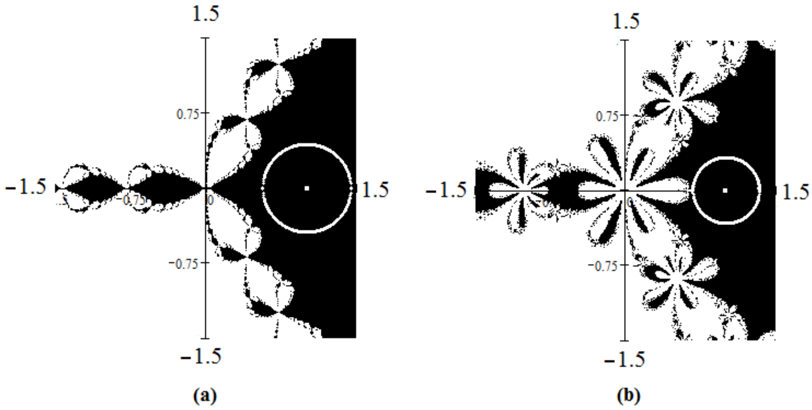

Fig. 2 Estimations of local convergence radii with proposed algorithm for the complex function $f(z) = z^3 - 1$: (a) For Newton method; (b) For One point Ezquerro-Hernandez method

Experiment 4

In this experiment we estimate the local radius of convergence of One point Ezquerro-Hernandez method using two relatively recent procedures proposed in [19] and [11] respectively. Recall that the first procedure gives the radius of convergence of the iteration $x_{n+1} = G(x_n)$ in terms of $q = \|G'(p)\|$ and of the Holder continuity constant K_t. More precisely, the radius of convergence is given by $r = [(1+t)(1-q)/K_t]^{1/t}$ (in our experiment $t = 1$). The second algorithm works as follows: Suppose that p is a solution of the equation $F(x) = 0$, there exists $F'(p)^{-1}$, $\|F'(p)^{-1}\| \leq \beta$ and F' is k-Lipschitz continuous on some $B(p, r_0) = \{x : \|x - p\| \leq r_0\}$. Let $\tilde{r} = \min\{r_0, r\}$, where $r = \zeta_0/[(1 + \zeta_0)\beta k]$ and ζ_0 is the positive real root of the equation $t^3 + 4t^2 - 8 = 0$. Then $\tilde{r}$ is a local radius of convergence. The mapping Ex.1 defined in Experiment 2, and the following two mappings are the test mappings:

$$Ex.2: \ F(x) = \begin{pmatrix} 3x_1^2 - x_1 x_2^3 + 3x_2 \\ 2x_1 + x_2^3 - 0.2x_2 \end{pmatrix},$$

$$Ex.3: \ F(x) = \begin{pmatrix} x_1 x_2^3 - x_1 + 2x_2^2 \\ x_1^2 + sin(x_2) \end{pmatrix}.$$

The results are given in Table 1. For reason of comparison, the third row contains the radii computed using our algorithm (maximal radii).

Method	Ex.1	Ex.2	Ex.3
Catinas algorithm	0.19241	0.15000	0.16223
Hernandez-Romero algorithm	0.22111	0.17027	0.13811
Our algorithm (maximal radius)	0.31923	0.25800	0.27415

Table 1: Local radii of convergence computed with different algorithms.

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